Finding Short Cycles in an Embedded Graph in Polynomial Time [1]

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ABSTRACT

Let $\mathcal{C}_1$ be the set of fundamental cycles of breadth-first-search trees in a graph G and $\mathcal{C}_2$ the set of the sums of two cycles in $\mathcal{C}_1$. Then we show that (1) $\mathcal{C} = \mathcal{C}_1 \bigcup \mathcal{C}_2$ contains a shortest Π-twosided cycle in a Π-embedded graph G;(2) $\mathcal{C}$ contains all the possible shortest even cycles in a graph G;(3) If a shortest cycle in a graph G is an odd cycle, then $\mathcal{C}$ contains all the shortest odd cycles in G. This implies the existence of a polynomially bounded algorithm to find a shortest $\Pi-$twosided cycle in an embedded graph and thus solves an open problem of B.Mohar and C.Thomassen[2,pp112]

Key Words $\Pi-$twosided cycle, breadth-first-search tree, embedded graph.

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1. Introduction

C.Thomassen showed that if cycles in a set of cycles satisfy the 3-*path-condition*, then there exists a polynomial time algorithm that finds a shortest cycle in this set[3]. As applications, he showed that the following types of shortest cycles may be found in polynomial time.

(1) *A shortest* $\Pi-$*noncontractible cycle in a* $\Pi-$*embedded graph;*
(2) *A shortest* $\Pi-$*nonseparating cycle in a* $\Pi-$*embedded graph;*
(3) *A shortest* $\Pi-$*onesided cycle in a* $\Pi-$ *embedded graph.*

But what about a family of cycles which do not satisfy the 3-path-condition? In their monograph[2, pp112], B.Mohar and C.Thomassen raised the following open problems:

(a) *Is there a polynomially bounded algorithm that finds a shortest* $\Pi-$*contractible cycle in a* $\Pi-$*embedded graph?*
(b) *Is there a polynomially bounded algorithm that finds a shortest* $\Pi-$*surface-separating cycle in a* $\Pi-$*embedded graph?*
(c) *Is there a polynomially bounded algorithm that finds a shortest* $\Pi-$*twosided cycle in a* $\Pi-$*embedded graph?*

Here in this paper we consider connected graphs and all the concepts used are standard following from [1,2].

Let G be a $\Pi-$embedded graph and T_x a *breadth-first-search tree* rooted at a vertex x of G. Let

$$\mathcal{C}_1 = \{C(T_x)|\forall x \in V(G), C(T_x)\ is\ a\ fundamental\ cycle\ of\ T_x\};$$

$$\mathcal{C}_2 = \{C|\exists x, y \in V(G), C = C(T_x) \oplus C(T_y)\}; \mathcal{C} = \mathcal{C}_1 \bigcup \mathcal{C}_2,$$

where "$\oplus$" is the operation defined as "$A \oplus B = (A - B) \cup (B - A)$" for any subsets A, B of $E(G)$.

Theorem A. *The collection* $\mathcal{C}$ *of cycles defined above satisfies the following conditions:*
(a). There exists a shortest $\Pi-$*twosided cycle (in a* $\Pi-$*embedded graph) in* $\mathcal{C}$*;*
(b). Each shortest even cycle of a graph is contained in $\mathcal{C}$*;*
(c). If a shortest cycle of a graph is an odd cycle, then every shortest odd cycle is contained in $\mathcal{C}_1$*.*

Therefore, Theorem A implies the existence of a polynomially bounded algorithm to find short cycles defined above.

Theorem B. *There exists a polynomially bounded algorithm to find a shortest* $\Pi-$*twosided cycle in a* $\Pi-$*embedded graph and all the shortest even cycles in a graph.*

This solves problem (c). Since every possible two-sided cycle in an embedded graph in the projective plane is contractible, we have the following

Corollary. *There is a polynomially bounded algorithm to find a shortest contractible cycle in an embedded graph in the projective plane.*

This answers the problems (a)-(c) in the case of projective plane graphs.

2. Proof of Main Result

A *generalized embedding scheme* of a graph G in a surface Σ is a set of a rotation systems $\pi = \{\pi_v | v \in V(G)\}$ together with a mapping $\lambda : E(G) \rightarrow \{-1, +1\}$,called a *signature*, where π_v is a clockwise ordering of edges incident with v. We define $\Pi = (\pi, \lambda)$ as an embedding scheme for the graph G. A cycle C of an embedded graph G ia called $\Pi-$twosided if C contains even number of edges with negative signature;otherwise, it is called $\Pi-$onesided.

For a vertex $v \in V(G)$, we may change the clockwise ordering to anticlockwise, i.e., π_v is replaced by its inverse π_v^{-1}, and $\lambda(e)$ is replaced by $-\lambda(e)$ for each edge e incident with v. Therefore, we obtain another embedding scheme $\Pi' = (\pi', \lambda')$. It is clear that a cycle is $\Pi-$twosided if and only if it is Π'-twosided. Two of such embedding schemes are *equivalent* if and only if one can be changed into another by a sequence of such *local changes*. Thus, for any spanning tree T of an embedded graph G, we may always assume that each edge e of T has signature $\lambda(e) = +1$ for convenience.

The key of the proof of Theorem A is the following classification of short cycles according to the distance between two vertices on such cycles. Let C be a shortest $\Pi-$twosided cycle in an embedded graph G. Then C must satisfy one of the following conditions:
(1).$\forall x, y \in V(C) \Rightarrow d_C(x, y) = d_G(x, y)$;
(2).$\exists x, y \in V(C) \Rightarrow d_C(x, y) > d_G(x, y$.

Remark: In the following, we will see that a cycle satisfying (2) may be written as a sum of two shorter cycles. Therefore, a shortest cycle in a

collection of cycles satisfying the 3-path-condition can't satisfy (2).

Lemma 1. *Let G and $\mathcal{C}$ be as defined in Theorem A and C a shortest* Π-*twosided cycle in G satisfying (2). Then $\mathcal{C}$ contains a shortest* $\Pi-$*twosided cycle of G.*

Proof of Lemma 1. We assume that C has a clockwise (anticlockwise)orientation $\overrightarrow{C}$ ($\overleftarrow{C}$)and for any two vertices u and v of C, $u\overrightarrow{C}v$ ($u\overleftarrow{C}v$) denotes the closed interval from u to v along $\overrightarrow{C}$ ($\overleftarrow{C}$). Similarly, we may define a segment uPv as the closed interval from u to v in a path P. If u is a vertex of C, then $u^-(u^+)$ is the predecessor (successor) of u along $\overrightarrow{C}$.

We consider two vertices x, y of C such that $d_G(x, y)$ is minimum subject to (2). Then for any shortest $(x - y)$ path P in G, P and C has no inner vertex on C(since otherwise, there would be another two vertices x_1, y_1 od C satisfying (2) and $D_G(x_1, y_1) < d_G(x, y)$). Let T_x be a breadth-first-search tree rooted at x. Then it contains a $(x - y)$ path P and each of the two cycles $P \cup x\overrightarrow{C}y$ and $P \cup x\overleftarrow{C}y$ is a $\Pi-$onesided cycle shorter than that of C and satisfies (1) (since otherwise one of them may be written as a sum of two shorter cycles,among then, one is a $\Pi-$twosided). Under these structure,each of $P \cup x\overrightarrow{C}y$ and $P \cup x\overleftarrow{C}y$ is a fundamental cycle of a breadth-first-search tree rooted at a vertex on itself. This completes the proof of Lemma 1.

Now we turn to the cycles satisfying (1). Firstly, the following result is easy to be verified and we omit the proof of it.

Lemma 2. *Let C be a shortest* $\Pi-$*twosided cycle in a* $\Pi-$*embedded graph G. Then for any three vertices x, u, v of C with*

$$d_c(x, u) = d_c(x, v) = [\frac{|C| - 1}{2}],$$

any shortest $(x - u)$path P and $x\overleftarrow{C}v$ has no inner vertex in common.

Thus, for a shortest $\Pi-$twosided cycle C satisfying the conditions in Lemma 2 and the last common vertex α (called *branched vertex*) contained in $(x - u)$path P_1 and $(x - v)$path P_2 of T_x, $x\overrightarrow{P_1}\alpha(= x\overrightarrow{P_2}\alpha)$ has no inner vertex on $x\overleftarrow{C}v(x\overrightarrow{C}u)$.

Lemma 3. *Let G and $\mathcal{C}$ be as defined in Theorem A and C a shortest* $\Pi-$*twosided cycle in G satisfying (1). Then $\mathcal{C}$ contains a shortest* $\Pi-$ *twosided cycle of G.*

Proof of Lemma 3.

Case 1 $|C| = 2n$, $n \in N$.

Let x and y be two vertices of C with $d_G(x, y) = n$ and T_x a breadth-first-search tree rooted at x whose edges are all assigned signature $\lambda = +1$. Then T_x contains a $(x - y)$path P, a $(x - y^-)$path P_1, and a $(x - y^+)$path P_2.

Subcase 1.1 Either $P_1 \subset P$ or $P_2 \subset P$.

We may suppose that $P_1 \subset P$ and $\lambda(y, y^+) = -1$ without generality. If P is not contained in C,then $P \cap C = \{x = x_1, x_2, ..., x_m = y\}$. By Lemma 2, we may choose an index i such that $C_i = x_i \overrightarrow{P} x_{i+1} \cup x_i \overrightarrow{C} x_{i+1}$ is an even $\Pi-$onesided cycle such that P has no inner vertex on $x_i \overrightarrow{C} x_{i+1}$ other than x_i and x_{i+1}. Since $|C_i| < |C|$, C_i can't be a sum of two shorter cycles. Therefore, C_i satisfies (1) and further, C_i is a fundamental cycle of a breadth-first-tree rooted at some vertex of C_i. Now $C_i \cup (P_2 \cup P \cup \{(y, y^+)\})$ has a shortest $\Pi-$twosided cycle in G.

If $P \subset C$, then P_2 can't be contained in C (since C is $\Pi-$twosided). As we have reasoned above, there exists a segment $x_j \overleftarrow{C} x_{j+1}$ of C such that $C_j = x_j \overleftarrow{C} x_{j+1} \cup x_j \overleftarrow{P_2} x_{j+1}$ is a fundamental cycle and so, $C_j \cup (P_2 \cup P \cup \{(y, y^+)\})$ contains a shortest $\Pi-$twosided cycle in G.

Subcase 1.2 $P_1 \not\subset P$ and $P_2 \not\subset P$.

If $\lambda(y, y^-) = +1$ or $\lambda(y, y^+) = +1$, then $P_1 \cup P \cup (y, y^-)$ or $P_2 \cup P \cup (y, y^+)$ will contain a shortest $\Pi-$twosided cycle. Assume further that $\lambda(y, y^-) = \lambda(y, y^+) = -1$. Then $P_1 \cup P_2 \cup \{(y, y^-), (y, y^+)\}$ also contains a shortest $\Pi-$twosided cycle in G.

Case 2. $|C| = 2n + 1$, $n \in N$

Let $x, y \in C$ with $d_C(x, y) = d_C(x, y^+) = n$ and T_x be a breadth-first-search tree rooted at x with all its edges been assigned $\lambda = +1$. Then T_x contains a $(x - y)$path P_1 and a (x, y^+)path P_2 and a branched vertex $\alpha \in P_1 \cap P_2$. Suppose that P_1 is not a part of C and $\lambda(y, y^+) = -1$. Then as we have shown in Case 1 there exists an index i such that $C_i = x_i \overrightarrow{C} x_{i+1} \cup x_i \overrightarrow{P_1} x_{i+1}$ is a fundamental cycle and thus $C_i \cup P_1 \cup P_2 \cup \{(y, y^+)\}$ contains a shortest $\Pi-$twosided cycle in $\mathcal{C}$. This completes the proof of Lemma 3 and so, finishes the proof of part (a) of Theorem A.

As for the proof of parts (b) and (c) of Theorem A, it follows from the proving procedure of (a) of Theorem A.

Based on the above statement, Theorem A is proved.

References

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